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Lidia Pawlusińska*

A Child's Mathematical Activity According to Professor Anna Zofia Krygowska[1]

Aktywność matematyczna dziecka w ujęciu profesor Anny Zofii Krygowskiej

Abstract: Leading Polish mathematics educator, Professor Anna Zofia Krygowska, devoted her entire life to searching for solutions that would help to overcome difficulties in learning mathematics.

Her academic activity contains many didactic proposals that can be successfully used in school practice to this day. This article focuses on one of these proposals – triggering students' mathematical activity. Actions specific to mathematical activity, that is, defining, mathematizing and schematizing, deducing and reducing, algorithmizing and noticing and using analogies can be successfully initiated in mathematical education at the first stage of education.

Keywords: mathematical activity, teaching mathematics, early school education, Anna Zofia Krygowska.

* Lidia Pawlusińska (ORCID: 0000-0003-0311-9755) – PhD, researcher and teacher at the Faculty of Social Sciences of the University of Szczecin, contact: lidia.pawlusinska@usz.edu.pl.

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Mathematical education is nothing else than developing a student's mathematical activity.
Anna Zofia Krygowska[2]

Teaching and learning mathematics continue to attract a wide range of people interested in the work of schools. The analysis of students' educational achievements in mathematics in international studies still gives an unsatisfactory picture of the quality of mathematics[3] education in the world. This also happens in our country, where the criticism of mathematics teaching also applies to early school education.[4] Where is the problem? Can we not find solutions to support students in developing mathematical competences? Or perhaps the requirements for the effects of mathematical education are changing like the proverbial ever-shifting kaleidoscope? Or maybe what has been heard in the public space so far has not been properly taken into account? The search for answers to these and many other questions in the field of mathematical education directs our attention to Professor Anna Zofia Krygowska's views about teaching the youngest students, that is, those who begin their adventure

2 M. Ciosek, *O wpływie Anny Zofii Krygowskiej na dydaktykę matematyki w świecie*, „Roczniki Polskiego Towarzystwa Matematycznego", seria V: „Dydaktyka Matematyki", 28 (2005) p. 147.

3 PISA – Program Międzynarodowej Oceny Umiejętności Uczniów; TIMSS – Międzynarodowe Badanie Wyników Nauczania Matematyki i Nauk Przyrodniczych.

4 E. Gruszczyk-Kolczyńska, *O kryzysie edukacji matematycznej dzieci. Rozpaczliwe wołanie o działania naprawcze*, „Matematyczna Edukacja Dzieci", 1 (2016) pp. 5–40; D. Klus Stańska, M. Nowicka, *Sensy i bezsensy edukacji wczesnoszkolnej*, Warszawa 2013; J. Filip, T. Rams, *Dziecko w świecie matematyki*, Kraków 2000; G. Treliński, *Integracja nauczania – uwarunkowania, praktyka*, in: *Matematyczna edukacja wczesnoszkolna. Teoria i praktyka*, eds. Z. Semadeni, E. Gruszczyk-Kolczyńska, G. Treliński, B. Bugajska-Jaszczot, M. Czajkowska, Kielce 2015, pp. 197–223; M. Dąbrowski, *Edukacyjna codzienność klasy trzeciej*, in: *Badanie umiejętności podstawowych uczniów klas trzecich klas szkoły podstawowej. Nauczyciel kształcenia zintegrowanego 2008 – wiele różnych światów?*, eds. M. Dagiel, M. Żytko, Warszawa 2009, pp. 125–175; A. Kalinowska, *Wyniki badań trzecioklasistów jako diagnoza kontekstów nauczania matematyki w klasach najmłodszych*, „Ruch Pedagogiczny", 2 (2014) pp. 147–156.

with mathematics by experiencing it in specific situations and using intuitions that are not burdened by learned mechanisms for solving mathematical tasks.[5]

When analysing Krygowska's views on teaching mathematics, I will focus on those aspects of her scientific and didactic activity that relate to school education, that is, those consciously transmitted by a teacher striving for the mathematical development of their students at the first stage of education. The article aims to show a child's mathematical activity as the aspect without which we cannot in any way decide about conscious and active learning. However, before we delve into Krygowska's views, I will describe this eminent scientist.

The profile of Anna Zofia Krygowska and her scientific and teaching activities

A great scientist, a great mathematician and a brilliant educator – these are only some of the terms that describe the activity of the creator of modern mathematics didactics, Professor Anna Zofia Krygowska (born on 19 September 1904 in Lviv, died on 16 May 1988 in Kraków). From the early years of her life, Krygowska was interested in mathematics and, despite her humanistic talents, she began studying mathematics at the Jagiellonian University (1923–1927). Having graduated, she worked as a mathematics teacher in Kraków schools. It was this professional period, as well as her previous experiences with secret teaching during World War II and tutoring during her secondary school years, that drew her attention to the difficulties others encountered when learning mathematics. Helena Siwek wrote about this period in Krygowska's life as follows:

> The experience she gained there made her a master at anticipating students' difficulties, planning interesting and deeply justified teaching proposals in mathematics and psychology, and reaching every student with mathematical knowledge.[6]

5 Z. Krygowska, *Międzynarodowa Konferencja w Sprawach Nauczania Matematyki w Krakowie*, „Roczniki Polskiego Towarzystwa Matematycznego", seria II: „Wiadomości Matematyczne", 4 (1961) No. 3, p. 291.

6 H. Siwek, *Profesor Krygowska – niezwykły człowiek, tytan pracy*, „Roczniki Polskiego Towarzystwa Matematycznego", seria V: „Dydaktyka Matematyki", 28 (2005) p. 85.

It should be noted that Krygowska was guided by practical experience throughout her entire scientific career. Her work focused on the concepts of teacher education, the development of research in the field of mathematics didactics and activities related to the reform of mathematics teaching in Poland. She published numerous articles in scientific and methodological journals, wrote conference reports and chapters of books for mathematics teachers as well as original and co-authored mathematics textbooks, popularized mathematics and views on its teaching during television lectures for teachers of secondary and primary schools, including teachers of early school education, promoted numerous doctoral theses in the field of mathematics didactics, as well as providing scientific supervision for five habilitation (post-doctoral) theses. All this proves her commitment to the continuous pursuit of effective solutions in the field of mathematics education. One of her many successes was the establishment of the Department of Mathematics Teaching (later the Department of Mathematics Didactics) in 1958, which she headed until her retirement in 1974. It was thanks to her efforts that mathematics didactics began to be perceived as a scientific discipline, and not as a 'weaker' relative of theoretical mathematics.[7] In the journal *Dydaktyka Matematyki* [Mathematics Didactics], Krygowska wrote about this important issue as follows:

> Mathematics didactics is a science that covers all issues related to learning and teaching mathematics. Today it is developing as an autonomous science, although research conducted in this field is largely interdisciplinary. Although the specificity of learning and teaching mathematics does not allow for their complete inclusion in any of the other developed disciplines, these problems appear and are considered at the intersection of such sciences, different in terms of subject and methodology, as mathematics, its methodology and history, psychology, computer science, cybernetics and linguistics. Solving such interdisciplinary issues requires integrating various research methods – from theoretical analyses to different types of empirical methods.[8]

7 Eadem, *Anna Zofia Krygowska – w stulecie urodzin*, „Matematyka", 6 (2004) pp. 4–11.

8 Z. Krygowska, *Główne problemy i kierunki badań współczesnej dydaktyki matematyki*, „Dydaktyka Matematyki", 1 (1982) p. 14.

Krygowska's scientific activity was also noticed outside Poland. In 1956, during a speech at the UNESCO conference in Geneva devoted to the modernization of mathematics teaching, the professor delivered a lecture on her concept of teaching mathematics, that is, the functional teaching of this subject. The concept was received with great enthusiasm, and Jean Piaget himself congratulated the speaker, emphasizing that her idea complied with his cognitive theory of human development.[9] Participation in international conferences confirmed Krygowska in her belief that the need to modernize mathematics teaching is consistent with the direction of global trends and the necessity of their implementation. The professor was convinced that the need she indicated for activating students in the process of learning mathematics was one of the priorities of these changes.[10]

Another important speech that Krygowska gave in the international arena was in Nice in 1970, when she was asked to give a lecture in the plenary session. She was the only Polish woman to be distinguished and honoured in this way. She then talked about a very important issue that, in her opinion, was the basis for success in teaching mathematics, that is, the education of future teachers of this subject. Each speech that Krygowska gave at international congresses and conferences was widely commented on in the world of science. These speeches touched upon the issues of developing students' mathematical activity (Moscow 1966), the importance of mathematical text in teaching (Lyon 1969), the selection of research problems in the field of mathematics didactics (Bayreuth 1971), the author's concept of mathematics didactics as a science (Exeter 1972), educating children and adolescents aged 10 to 16 (Karlsruhe 1976), teaching geometry (Congress of Belgian Teachers 1961), the role of calculus in teaching mathematics (Arlon 1968), as well as developing mathematical reasoning and identifying problems in teaching mathematics (Knokke 1970, 1972). She also focused on the modernization of primary education (Budapest 1962) and mathematics teaching in primary school (Visegrád 1974) and many others.[11] In her activity, there were important periods when she worked in foreign scientific institutions,

9 M. Ciosek, *O wpływie Anny Zofii Krygowskiej*.

10 Ibidem.

11 Ibidem, pp. 131–135.

provided care to foreign interns coming to Kraków, collaborated with foreign journals (*Educational Studies in Mathematics* or *Recherches en Didactique des Mathematiques*) and published articles in foreign languages (55 works in English, Czech, French, Spanish, Japanese, German, Russian, Romanian, Hungarian and Italian). An important element of her scientific and didactic activity was her cooperation with the International Commission for the Study and Improvement of the Teaching of Mathematics (CIEAEM), which began in 1958. The commission's work aims to improve mathematics teaching. Its main activity is organizing conferences at which mathematics teachers, pedagogues, psychologists and mathematics didacticians from various parts of the world share the results of their work. The latest CIEAEM conference was held in August 2023 in Malmö, Sweden. This time, the meeting aimed to discuss mathematical practices in teaching and educational mathematical practices in connection with other practices, including vocational education, art and current social phenomena.[12]

Krygowska was an extremely valued person in Poland and abroad. Scholars from around the world whose works touched upon the topic of learning and teaching mathematics emphasized her extraordinary commitment and professionalism. Professor Josette Adda wrote that Krygowska made teaching her passion and was able to pass this passion on to others. Professor Henri Wermus believed that the Polish didactician was an example of a great pedagogue and also a personality endowed with authentic humanism. In the foreword to his book *Mathematics as an Educational Task*, Professor Hans Freudenthal expressed his thanks to Professor Krygowska for everything he had the opportunity to learn from her.[13]

Students' activities in the process of learning mathematics

In the 1970s, Professor Krygowska wrote, 'Today we want not only to teach mathematics, but also to educate through mathematics.'[14] She understood education through mathematics as the need to use specific features of mathemat-

12 Strona CIEAEM, https://sv-se.eu.invajo.com/event/cieaem/cieaem74, accessed: 15.03.2023.

13 M. Ciosek, *O wpływie Anny Zofii Krygowskiej*.

14 Z. Krygowska, *Zarys dydaktyki matematyki*, part 1, Warszawa 1979, p. 13.

ics to intellectualize the attitudes of a young person so that they could consciously use theoretical thinking to solve mathematical problems and those occurring outside mathematics. She stressed the need to develop the numerical, dimensional and spatial intuitions necessary for a comprehensive understanding of the surrounding reality. She saw the need to develop skills in using the language of mathematics and knowledge of mathematics-specific methods that are widely used in the world of technology, and without knowledge of which the use of these technologies would be difficult. She strongly emphasized the need to master mathematics learning techniques and use various mathematical sources, necessary – in her opinion – to develop general learning techniques because these skills are indispensable in an era in which 'continuous learning is a form of being human'. Finally, she believed that mathematical education at school, although limited in terms of content, should not bar the way to further mathematical education, as long as it was consistent with the student's life plans.[15] While justifying the need for mathematical education in this way, she also defined mathematics teaching not as the transmission of *a priori* accepted teaching content or the mastery of mathematical procedures laboriously developed during school classes, but as a field whose experience during learning provides wide possibilities for its application – above all, it enables the transfer to other areas of knowledge. She clearly emphasized that teaching should be understood as controlling the learning of mathematics.[16]

Looking for an answer to the question about the shape of mathematical teaching, Krygowska formulated three levels of goals that, in her opinion, determine the shape of mathematics education. The first level covers basic knowledge and skills in the field of mathematics, that is, those that can currently be found in the core curricula of general education for a given educational stage. It is nothing more than teaching content formulated in the language of requirements, for example, students compare numbers, multiply and divide by memory numbers included in the multiplication tables, describe solutions to mathematics problems using operations, equations with boxes, drawings or in another way of their choice, use the signs: <, =, >, and calculate the perimeters

15 Eadem, *Koncepcje powszechnego matematycznego kształcenia*, Kraków 1981, pp. 48–49.

16 Eadem, *Zarys dydaktyki matematyki*, part 2, Warszawa 1977, p. 14.

of rectangles.[17] The second level includes attitudes and behaviours appropriate for mathematical activity and a certain awareness of some elements of mathematical methodology, such as an active attitude towards mathematical problems and the disposition to notice and formulate them, the ability to use certain strategies when solving problems, a certain level of geometrically organized spatial imagination, an active attitude in the search for justifications and understanding their meaning. It also includes those explicitly included in the core curriculum, such as developing the skills of critical and logical thinking, reasoning, argumentation and inference.[18] Attitudes and behaviours appropriate for mathematical activity are developed (and should be developed) gradually in the course of learning the content and shaping of the skills specified in the core curriculum. The third and highest level of goals in the hierarchy includes intellectual attitudes and behaviours functioning outside mathematical activity, developed by transferring and adapting attitudes and specific behaviours to other areas of human activity.[19] Krygowska considered achieving this level to be the highest goal of teaching mathematics. Indicating subsequent levels of 'initiation' into the essence of mathematical education, she warned against focusing too much on achieving the first-level goals. She pointed out that acquiring knowledge and developing skills are not tantamount to shaping the attitudes and behaviours appropriate for mathematical activity, and certainly not tantamount to achieving the third-level goals.[20]

However, even the loftiest goals do not guarantee success if students are not committed to achieving them. Krygowska convinced us that the principle of active and conscious participation of students in the teaching process is the guiding principle and a necessary condition for the implementation of other

17 The Regulation of the Minister of National Education of 14 February 2017 on the pre-primary curriculum and the general education school curricula in primary schools, including the curricula for pupils with moderate and severe intellectual disabilities, and for general education in stage 1 sectoral vocational schools, general education in special schools preparing for employment and general education in post-secondary schools (Journal of Laws of 2017, item 356).

18 Ibidem.

19 Eadem, *Elementy aktywności matematycznej, które powinny odgrywać znaczącą rolę w matematyce dla wszystkich*, „Dydaktyka Matematyki", 6 (1986) pp. 25–41.

20 Ibidem, p. 27.

teaching principles[21] formulated in pedagogical theories. It is the principle of conscious and active participation that guarantees that learners will consciously trigger their mathematical activity with the full involvement of their imagination and thinking and full understanding of the subsequent concrete, imaginary and mental activities. And this is what Krygowska considered to be among the most important goals of teaching mathematics.[22] She emphasized that:

> … the more rationally we organize the illustrative basis of elementary mathematics from the point of view of mathematics and psychology, the more correctly we direct this illustrativeness towards proper mathematization and the more fully and easily we activate the student's mathematical imagination and mathematical thinking.[23]

At the same time, she pointed out that, adopting the leading principle of a student's conscious and active participation in the teaching process, we focus our attention on shaping mathematics learning techniques. It is learning mathematics that makes a student's activity organized.[24]

Among the activities that students undertake while learning mathematics, Krygowska detailed: acquiring and assimilating mathematical knowledge, for example, in the form of lectures, mathematical texts, discussions or mathematical films; practicing elementary mathematical skills; solving tasks using mastered mathematical methods; editing mathematical texts; writing down notes, illustrating task situations with diagrams; and developing skills in using mathematical language. She paid particular attention to the 'creative activity' category. She believed that students are creative in mathematics lessons when they notice and formulate problems, construct and define concepts, subjectively discover new theorems, as well as generalize and use mathematics to solve problems from other fields, including in non-standard situations.[25] Krygowska strongly linked this category with important mathematical

21 Z. Krygowska, *Zarys dydaktyki matematyki*, part 2, p. 3.
22 Ibidem.
23 Ibidem.
24 Ibidem.
25 Ibidem, p. 14.

activities and indicated them as provocable at every level of teaching. Among these, she distinguished: noticing and using analogies, schematizing, defining concepts, interpreting and rationally using definitions, deducing and reducing, coding, constructing and rationally using symbolic language, algorithmizing and rationally using algorithms.[26] She called these mathematical activities and warned against their mutual isolation. She argued that in the course of a student's creative work, these activities interpenetrate and often complement each other, which is their undoubted advantage.

A child's mathematical activities

Noticing and using analogies

In the opinions of mathematicians, analogy and its use are an important mechanism of mathematical creativity – this is what Krygowska claimed, referring to the words of one of the most outstanding Polish mathematicians, Stefan Banach. She added that mathematicians constantly use analogy at every level of their considerations, that is, at the intuitive level within emerging ideas and at the formal level when they identify structures based on morphisms.[27] The universality of analogies in mathematical activities, and also in other fields, is unquestionable. Analogy should therefore appear in mathematics teaching as a tool for thinking, exploring and enabling students to organize in their minds what they perceive in the surrounding reality.

Many attempts to define analogy have been made not only in mathematics. Initially, it was defined as a proportion, that is, the equality of the ratios $a : b = c : d$, when we can determine the fourth quantity based on the three given ones.[28] Analogy is (from Greek) similarity, correspondence. Considering the etymological origin of the term, George Polya pointed out that it is a certain type of similarity, that is, one that allows the identification of certain rela-

26 Z. Krygowska, *Elementy aktywności matematycznej.*

27 Ibidem.

28 J. Konior, *Wnioskowanie przez analogię i potrzeba jego rozwijania w edukacji matematycznej*, „Nauczyciel i Szkoła", 1–2 (2004) p. 71.

tionships between elements of the objects under consideration.[29] Jan Filip and Tadeusz Rams added that in mathematics we can consider analogy in several situations, for example, when we notice the same relations between the elements of the considered objects (the relation of parallel sides in a square and edges in a cube), and when we indicate a mapping preserving certain relations between the considered objects (the inclusion relation between a square and a rectangle and between a cube and a cuboid).[30]

Considering analogies in the context of school situations at the first stage of education, it can be said that students use them when, based on a mathematical fact that is sufficiently known to them, they make decisions about another fact that is similar to it. In order for them to do this, they must know the features of the former, notice them in the latter and specify in the former those properties that they intend to attribute to the latter (e.g., when students, completing addition tables, are asked to complete subtraction tables). Krygowska emphasized that teaching mathematics would be impossible without the use of analogies from the very beginning:[31]

> Generalizations and great syntheses, so characteristic of modern mathematics, are analogies expressed in mathematical language. The sense of analogies plays an essential role in developing our intuition in mathematics.[32]

Krygowska recommended analogy as an activity that also opens up extensive opportunities for weaker students. She noticed that the simple similarities and associations they use can constitute the basis for the formation of elementary concepts. Moreover, the search for what changes in certain transformations of the considered objects leads to structural analogy, and the analogy of procedures can be used to develop the ability to consciously use mathematical methods.[33] The scholar wrote:

29 G. Polya, *How to Solve It: A New Aspect of Mathematical Method*, Princeton 2014.

30 J. Filip, T. Rams, *Dziecko w świecie matematyki*, s. 36–37.

31 ibidem.

32 Z. Krygowska, *Zarys dydaktyki matematyki*, part 1, p. 13.

33 Eadem, *Elementy aktywności matematycznej*, p. 31.

> … at every step, various activities of students can be focused on each element of the teaching content, aimed at perceiving a specific analogy that is important in a given situation, at transferring properties from one situation to another situation perceived as analogous, at generalizations, at hypotheses and at perceiving new problems, etc.[34]

She also emphatically claimed that in order to develop students' sense of analogy, we should stop demonstrating analogies to them through examples. What is necessary here is to organize classes in a way that encourages students to see analogies and use them.[35]

Mathematization and schematization

Krygowska considered mathematization to be one of the main mathematical activities. It involves organizing everything that surrounds us using mathematical means. Mathematization is also seen as a process responsible for the 'transformation' of perceived reality into the abstract world of mathematics,[36] and the ability to use it is described as a universal human ability, such as the ability to speak or to represent something by drawing it.[37] This approach to mathematization indicates the importance it should be given in school teaching, especially since introducing students to the use of mathematical methods is an important goal of school education.

We can find opportunities to initiate this process in every lesson, for example, when students look for an answer to a question posed in a text problem or plan the arrangement of desks in the room for a grandparents' day celebration. An important feature of mathematization is that, in addition to the ability to obtain answers in specific situations, it also provides the perspective of seeing the structure of all mathematics, organizing the elements of theory and assess-

34 Ibidem, p. 3.

35 Ibidem.

36 G. Treliński, *Matematyzowanie jako składowa kompetencji matematycznej*, „Matematyczna Edukacja Dzieci", 1 (2016) p. 67.

37 A. W. Bell, *The Learning of Process Aspects of Mathematics*, „Educational Studies in Mathematics", 10 (1979) pp. 361–387, DOI: 10.1007/BF00314662.

ing the effects of actions taken and chosen directions of exploration.[38] Given the role assigned to mathematization, it should be made a permanent, planned element of school teaching.

Krygowska defined mathematization in teaching as 'the construction of a mathematical schema for some system of relations, captured by the analysis of a real, imagined or already abstract situation, or specified in another field of concepts, e.g., in another science'[39] or 'the construction of a still semi-illustrative thought pattern, which could later be transformed and incorporated into an already complete mathematical schema'.[40]

In the first case, students search for a mathematical schema in a mathematical theory known to them. The relations they notice are presented in a schema that could be directly incorporated into this theory and, above all, are considered within this theory using the means available in it. We then say that students have built a mathematical schema. However, in the second case, students base their construction on a semi-illustrative thought pattern that was not based on any mathematical theory, is not a mathematical schema and cannot be directly incorporated into any theory. Krygowska called this stage preliminary, primitive or illustrative mathematization. Nevertheless, she made it extremely important for the construction of students' mathematical knowledge, provided that, from the very beginning, the teacher properly oriented it towards later mathematization. She warned that improper planning of the teaching process leading to the construction of initial schemas (thought patterns) is the source of many subsequent mathematical difficulties for students.[41] I fully agree with Krygowska's view that 'students' experiences and intuitions should be mathematized as early as possible, as radically as possible, and from the very beginning, purely from the point of view of mathematics, but always in a natural way'.[42]

Mathematization is inextricably linked with schematization, which is identified as one of its mechanisms. It involves isolating facts, that is, important fea-

38 G. Treliński, *Matematyzowanie jako składowa.*

39 Z. Krygowska, *Zarys dydaktyki matematyki*, part 1, p. 48.

40 Ibidem, p. 48.

41 Ibidem, p. 49.

42 Ibidem, p. 79.

tures, from a certain structure, based on which we construct a schema using the language of mathematics. Created out of the need to simplify the situation under consideration and dependent on the degree of accuracy adopted *a priori*, the schema takes various forms, fulfils various functions and may have different meanings in school practice.

We often encounter schematization in early school education. Most often, we refer to material schemas in the form of a drawing, graph or physical model cut out from a piece of paper. In such situations, students' thinking activities focus on isolating a specific structure from the wealth of structures relevant to the situation under study.[43] This directs our attention to the fact that schematization does not occur in isolation from other mathematical activities, for example, analogy, without which the construction of a schema would be impossible. It is the perception of similarities in mathematical objects and the relationships between them that facilitates the construction of a schema and – what is more – enables the development of mathematical knowledge as a coherent structure.

Coding is another activity closely related to schematizing because, without an appropriate code adopted when constructing a schema, we will not obtain an appropriate representation of the situation under study. Of course, schematization does not direct our efforts to building schemas assigned to the specific situation under consideration. Teaching efforts should be aimed at directing students' attention to the possibilities of applying the constructed schema in various situations. This happens when the situation described with a schema, through analogy and properly applied code, makes the resulting schema more general, aspiring to be called an algorithm. It is in this aspect that Krygowska pointed to multivalence as a constitutive feature of a schema. However, to convince students to understand the role of a schema in this way, an appropriate number of experiences are needed, experiences that end with the independent construction of schemas and the search for analogous situations in which they can also be used. This is a very important process that naturally directs students' thoughts towards generalization.[44]

43 Z. Krygowska, *Elementy aktywności matematycznej*.

44 Ibidem.

Defining, interpreting and rationally using a definition

Mathematics has a certain set of concepts with which it describes its achievements. It is similar in other scientific fields. However, there is a difference in the nature of these concepts, which, unfortunately, is related to the greatest barrier that appears when learning mathematics, that is, abstractness. The lack of real-world equivalents makes it difficult for students to understand these concepts and places high demands on teachers in planning, developing and implementing mathematics lessons. Krygowska indicated two ways of introducing mathematical concepts, which involve different approaches to teaching. She did not deny any of them but only emphasized that they initiate various cognitive activities of students.[45] The first way is for a teacher or textbook to present a new definition. Here, a student's fundamental activity is to informalize a verbal or symbolic notation and provide examples or counterexamples. The second one involves the organization of cognitive activity in which, with the teacher's discreet support, students notice a certain fact and formulate its definition. The feature that distinguishes these ways is the approach to shaping them. By choosing the first one, students understand concepts based on the interpretation of the fact described, in the second case, the understanding of the concept precedes its definition.

A definition is understood as a certain convention which, formulated based on a given theory and the syntactic rules operating within it, allows us to replace one notation with another. Krygowska indicated this replacement, which is called constructing definitions, as an activity rich in various intellectual procedures, such as comparison, abstraction, generalization, analysis and synthesis. She considered the correct interpretation and rational use of definitions to be equally important in the process of developing concepts. She indicated these two activities as opportunities to build a bridge between a child's existing mathematical intuitions, images of an undefined concept and the formal requirements of mathematics. She considered all these activities to be necessary in school practice.[46]

45 Eadem, *Zarys dydaktyki matematyki*, part 2, p. 79.

46 Eadem, *Elementy aktywności matematycznej*.

By concentrating on the didactic aspect of constructing, interpreting and rationally applying a definition, we will focus on its semantic meaning, that is, the description of a given object acceptable for what a student wants to define.[47] Some of the first attempts to characterize mathematical objects or facts occur in the early school period. However, at this stage, we cannot expect mathematical concepts to be defined in the strict sense of the word. This is the beginning of the adventure with school mathematics, when, according to Krygowska, we can expect students to give an explanation or definitional description and this is what we should demand.[48]

Children's first attempts at defining concepts are expressed in everyday language and filled with previously learned mathematical terms. These are often imprecise and usually focus on the features that the student noticed in the previously considered model of a given concept or in what they used it for. Krygowska called such descriptions and explanations primitive or direct. In addition to these, at the next level of 'initiation' in the definition process, she distinguished generalizing description, that is, taking into account the change of constants, and constructive description, which involves modifying the initial idea and unclear description towards the definitional approach to a given concept. She indicated all of these as necessary stages of the long path leading to a formal definition.[49]

In the context of early school education, attention is drawn to a description or definitional explanation, still embedded in a certain universe, which is strongly limited by children's linguistic resources and previously acquired rules. Nevertheless, when defining an object or a relation, students make corrections in subsequent versions, and their formulations become clearer and more precise. This indicates that the definition process is not only about assigning meaning to mathematical terms but is also closely related to subsequent in-depth analyses of a given concept.[50]

47 Eadem, *Zarys dydaktyki matematyki*, part 3, Warszawa 1977, p. 95.

48 Ibidem, p. 94.

49 Ibidem, pp. 94–97.

50 Eadem, *Elementy aktywności matematycznej*.

Deducing and reducing

Deduction, which in mathematics plays a fundamental role both in research activities and at the stage of verifying research results, is available to students at the first stage of education to a very limited extent. Krygowska therefore found it inappropriate to construct a lesson as a deductive argument from theories sanctioned in mathematics. She argued that students are too immature to fully understand the nature of the deductive method and its role in mathematics. Instead, she proposed tasks that would support the development of students' deductive reasoning. However, even here, as she wrote, one should not resort to a strict deduction with its rigour of formalities. At this stage, there is a need for deduction that is looser and adapted to the level of teaching, which will gradually familiarize students with the activity of deduction from the very beginning.[51]

However, there is no mathematical activity without reasoning, the main role of which is to correctly draw conclusions from the analyses performed. This activity, characteristic of learning mathematics, should be initiated in early school education. For example, students use deductive reasoning when solving word problems. However, this does not involve drawing valid conclusions from assumptions considered to be true. Most often, in school practice, we come across conclusions drawn based on students' observations and experiences they have conducted. This is how, at the first stage of education, children can construct the first hypotheses based on reasoning, leading them from what they know to facts or information that are new to them. Reduction is another, equally important type of reasoning.[52] Here, the truthfulness of the conclusion from which we start reasoning may lead to a false reason. Reductive reasoning is called probable reasoning and, when using it, we must be convinced that reasons other than the chosen ones are unlikely. Inductive reasoning is a special case of reductive reasoning.[53] Inductive reasoning begins with the observation of certain facts, based on which we draw a general

51 Ibidem.

52 J. M. Bocheński, *Współczesne metody myślenia*, Poznań, 1992, p. 78.

53 Ibidem.

conclusion about the facts observed. As in reductive reasoning, here we cannot talk about the absolute truthfulness of the inferences because we rely on our own experience and observations.

Krygowska distinguished three types of reasoning that can be applied in teaching mathematics. These are: empirical reasoning, intuitive reasoning and formal reasoning.[54] In formal reasoning, we rely on full awareness of the adopted basis of deduction, and each conclusion is drawn precisely based on previously known theorems and definitions. In formal reasoning, deduction is supported by reduction.[55] This type of reasoning is extremely advanced and we rarely encounter it in early school education. Krygowska pointed out empirical and intuitive reasoning to be appropriate at the first stage of education. We make empirical inferences based on experiments carried out in the material world. Observed facts supported by inductive reasoning enable us to formulate observations in the form of mathematical hypotheses.[56] Intuitive reasoning, involving images of concepts often not supported by formal definitions, is used to reason about the situation being examined.[57] Each of these types of reasoning is connected with the need to take into account the psychological process of interiorization in the teaching of mathematics, which is so important at the beginning of school education.

Coding, constructing and rationally using symbolic language

The language of mathematics is very rich and shaped not only by words. It contains images, symbols, abstract concepts and sentence structures typical of mathematical logic. All this creates a huge space for children's activity, a space in which properly organized education supports the development of linguistic competences in mathematics. Therefore, there is no need to emphasize that diversity and a multitude of experiences are a *sine qua non* condition for a child's linguistic richness.

54 Z. Krygowska, *Zarys dydaktyki matematyki*, part 1, p. 137.

55 Eadem, *Elementy aktywności matematycznej*.

56 Eadem, *Zarys dydaktyki matematyki*, part 1, p. 137.

57 Ibidem, p. 144.

We organize the experience of mathematics in a child's education in many ways. We can refer, for example, to a game in which properly selected teaching resources will allow the child, through interiorization, to define concepts, generalize, formulate first conclusions or notice relations in order to then verify them and look for appropriate applications. All this contributes to transforming the child's colloquial language into the formal language of mathematics. Organizing situations that inspire the child to act should also take into account the development of language competences.

Krygowska stressed the importance of symbolism that fills the language of mathematics. She believed that the activity of coding, constructing symbols and the ability to use them should be developed in teaching.[58] At the same time, she warned against actions that would result in the loss of the semantic meaning of mathematical symbols. She saw such a danger in situations where a symbol precedes a concept. According to her, a symbol should be introduced simultaneously with the concept to which it belongs, or after its introduction. We also cannot allow situations in which a symbol replaces a concept, for example, when the concept of the perimeter of a rectangle is introduced as the formula (2a + 2b). Krygowska emphasized the operational nature of the language of mathematics and she balanced its correct development with the development of correct reasoning. She considered partial and gradual formalization of the language as introducing students to the mathematical method, that is, implementation of the most important goal in teaching this subject.[59]

Algorithmization and the rational use of algorithms

We refer to an algorithm in the context of teaching mathematics after Krygowska as a finite sequence of basic activities, the reliable execution of which guarantees obtaining the correct solution.[60] We understand algorithmization as the process of constructing an algorithm that requires many mental activities from a student. According to the scholar, the operational nature

58 Eadem, *Elementy aktywności matematycznej*.

59 Eadem, *Zarys dydaktyki matematyki*, part 2, p. 104.

60 Eadem, *Zarys dydaktyki matematyki*, part 1, pp. 114–115.

of mathematics and the openness of a child's mind to the operationalization of this field of knowledge favour the use of algorithms in teaching from an early age. She also suggested organizing the teaching process so as to make students construct algorithms themselves as plans for solving certain tasks and to also use ready-made algorithms rationally.[61]

The use of algorithms in teaching mathematics is indisputable. They appear wherever students use previously developed methods for solving a given type of tasks. They make students' work easier and faster. However, they may pose a certain threat. Namely, they may cause mathematics to be perceived as a field consisting of a set of procedures to be mastered by heart. For this reason, Krygowska considered it desirable in teaching to introduce students to the rational use of algorithms, taking into account a conceptual approach to a given issue.[62] Of course, she did not deny the use of ready-made algorithms. She appreciated their role as a tool for certain automation in the field of, for example, accounting skills, which allows a teacher to allocate time for the students' creative activities. However, she warned against using only algorithms, as this is a great simplification in teaching, contributing to the mindless repetition of practiced steps.[63]

Constructing algorithms also triggers other mental activities. This skill is much more difficult than the previous one. We start by precisely formulating a solution plan, which is the equivalent of the steps leading to achieving the result. This important stage in searching for the final shape of the algorithm involves reorganizing the knowledge acquired so far and extracting the elements from it that are important for a given solution. These activities require precise and logical thinking, and these skills only start to be developed at the beginning of mathematical education. The first constructions of algorithms are therefore initially based on intuitive reasoning and, with an appropriate number of attempts, lead students to create a functional plan for the algorithm. Only at subsequent stages of education, in connection with changing data, analysing possible solution strategies and using formal reasoning, do we obtain the algorithm that Krygowska considered to be unambigu-

61 Eadem, *Elementy aktywności matematycznej*.

62 Eadem, *Zarys dydaktyki matematyki*, part 1, pp. 114–115.

63 Eadem, *Elementy aktywności matematycznej*.

ous, general and effective. All these activities lead students to the formalization of the mathematical issues selected in the process of constructing algorithms, an important stage of which is the construction of algorithms by students at the first stage of education.[64]

In algorithmic teaching, it is a mistake to either use algorithms or concentrate students' efforts on the construction of algorithms. Krygowska clearly indicated that there is a place for both in mathematics teaching. She encouraged teachers to provoke situations in which students then compare algorithms, look for their simpler forms, create algorithms analogous to the given one, assess the correctness of algorithms and write them down in various forms (tree diagrams, organigrams, etc.).[65]

* * *

Generalizing the considerations regarding mathematical activities, it can certainly be said that they should be initiated at the stage of early school education. This forces the organization of the school process of a child's mathematical education in such a way that, in addition to acquiring knowledge and developing individual skills, the school classroom becomes a space for experiencing attitudes and intellectual techniques specific to mathematical activities. According to Krygowska, these are responsible for the development of more general techniques and attitudes in each of us and enable the transfer of mathematical knowledge to other fields.[66] They shape our views on the role of science and theoretical thinking in a world of constant economic, civilizational and cultural changes.

Teaching mathematics is intended to enable students to fully engage in the cognitive process. It cannot be associated with students reproducing laboriously developed operating procedures. This triggering of students' mathematical activities frees lessons from 'adult mathematics'. It allows students to create their own mathematics, independently shape concepts, notice and describe

64 Eadem, *Zarys dydaktyki matematyki*, part 1, pp. 114–115.

65 Ibidem, p. 116–118.

66 Ibidem.

relations between mathematical objects, and find their own path to mathematics. Krygowska's main message was to make everyone who teaches mathematics aware that acting in the world of abstraction consists of a system of basic, specific mental operations. This is the basis for conscious and planned teaching, and the mathematical activities provoked should serve a deeper and more complete perception of mathematics.

Streszczenie: Wybitna polska dydaktyk matematyki profesor Anna Zofia Krygowska całe swoje życie poświęciła poszukiwaniu rozwiązań ułatwiających pokonywanie trudności w uczeniu się matematyki. W jej działalności naukowej odnajdziemy wiele propozycji dydaktycznych, które do dziś możemy z powodzeniem stosować w praktyce szkolnej. W niniejszym artykule skoncentrowano się na jednej z tych propozycji, czyli wyzwalaniu aktywności matematycznej uczniów. Specyficzne dla działalności matematycznej aktywności, tj. definiowanie, matematyzowanie i schematyzowanie, dedukowanie i redukowanie, algorytmizowanie czy dostrzeganie i wykorzystywanie analogii z powodzeniem możemy inicjować w edukacji matematycznej na pierwszym etapie edukacyjnym.

Słowa kluczowe: aktywność matematyczna, nauczanie matematyki, edukacja wczesnoszkolna, Anna Zofia Krygowska.

Bibliography

Works by Zofia Krygowska

Krygowska Z., *Elementy aktywności matematycznej, które powinny odgrywać znaczącą rolę w matematyce dla wszystkich*, „Dydaktyka Matematyki", 6 (1986) pp. 25–41.

Krygowska Z., *Główne problemy i kierunki badań współczesnej dydaktyki matematyki*, „Dydaktyka Matematyki", 1 (1982) pp. 7–60.

Krygowska Z., *Koncepcje powszechnego matematycznego kształcenia*, Kraków 1981.

Krygowska Z., *Międzynarodowa Konferencja w Sprawach Nauczania Matematyki w Krakowie*, „Roczniki Polskiego Towarzystwa Matematycznego", seria II: „Wiadomości Matematyczne", 4 (1961) No. 3, pp. 289–293.

Krygowska Z., *Zarys dydaktyki matematyki*, part 1, Warszawa 1979.

Krygowska Z., *Zarys dydaktyki matematyki*, part 2, Warszawa 1977.

Krygowska Z., *Zarys dydaktyki matematyki*, part 3, Warszawa 1977.

Literature

Bell A. W., *The Learning of Process Aspects of Mathematics*, „Educational Studies in Mathematics" 10 (1979) pp. 361–387, DOI: 10.1007/BF00314662.

Bocheński J. M., *Współczesne metody myślenia*, Poznań, 1992.

Ciosek M., *O wpływie Anny Zofii Krygowskiej na dydaktykę matematyki w świecie*, „Roczniki Polskiego Towarzystwa Matematycznego", seria V: „Dydaktyka Matematyki", 28 (2005) pp. 129–154.

Dąbrowski M., *Edukacyjna codzienność klasy trzeciej*, in: *Badanie umiejętności podstawowych uczniów klas trzecich klas szkoły podstawowej. Nauczyciel kształcenia zintegrowanego 2008 – wiele różnych światów?*, eds. M. Dagiel, M. Żytko, Warszawa 2009, pp. 125–175.

Filip J., Rams T., *Dziecko w świecie matematyki*, Kraków 2000.

Gruszczyk-Kolczyńska E., *O kryzysie edukacji matematycznej dzieci. Rozpaczliwe wołanie o działania naprawcze*, „Matematyczna Edukacja Dzieci", 1 (2016) pp. 5–40.

Kalinowska A., *Wyniki badań trzecioklasistów jako diagnoza kontekstów nauczania matematyki w klasach najmłodszych*, „Ruch Pedagogiczny", 2 (2014) pp. 147–156.

Klus Stańska D., Nowicka M., *Sensy i bezsensy edukacji wczesnoszkolnej*, Warszawa 2013.

Konior J., *Wnioskowanie przez analogię i potrzeba jego rozwijania w edukacji matematycznej*, „Nauczyciel i Szkoła", 1–2 (2004) pp. 67–77.

Polya G., *How to Solve It: A New Aspect of Mathematical Method*, Princeton 2014.

The Regulation of the Minister of National Education of 14 February 2017 on the preprimary curriculum and the general education school curricula in primary schools, including the curricula for pupils with moderate and severe intellectual disabilities, and for general education in stage 1 sectoral vocational schools, general education in special schools preparing for employment and general education in post-secondary schools (Journal of Laws of 2017, item 356).

Siwek H., *Anna Zofia Krygowska –w stulecie urodzin*, „Matematyka", 6 (2004) pp. 4–11.

Siwek H., *Profesor Krygowska – niezwykły człowiek, tytan pracy*, „Roczniki Polskiego Towarzystwa Matematycznego", seria V: „Dydaktyka Matematyki", 28 (2005) pp. 83–98.

Treliński G., *Integracja nauczania – uwarunkowania, praktyka*, in: *Matematyczna edukacja wczesnoszkolna. Teoria i praktyka Z. Semadeni*, eds. E. Gruszczyk-Kolczyńska, G. Treliński, B. Bugajska-Jaszczot, M. Czajkowska, Kielce 2015, pp. 197–223.

Treliński G., *Matematyzowanie jako składowa kompetencji matematycznej*, „Matematyczna Edukacja Dzieci", 1 (2016) pp. 65–82.